Problem Set from Past Spring 2025

Math Olympiad Club Zurich

Fall 2025

Problem A-2 (IMC 1999)

Does there exist a bijective map $\pi\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ such that

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} < \infty?$$

Problem 2 (IMC 1994)

Let $f \in C^1\left(]a,b[\,,\mathbb{R}\right)$ with $\lim_{x\to a^+} f(x) = +\infty$, $\lim_{x\to b^-} f(x) = -\infty$, and $f'(x) + f^2(x) \geq -1$ for all $x \in\,]a,b[$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

Problem 3 Bernoulli Competition 2024

Suppose $x_0 \in \mathbb{R}$ and $x_{n+1} = \sum_{i=0}^{n} (-1)^i \sin(x_i)$ for $n \geq 1$.
1) What is the range for the x_0 such that $\lim_{n\to\infty} x_n$ exists? What is the value of the limit depending on x_0 in the range?
2) Suppose $x_0 = 1/4$. Find $\lim_{n\to\infty} \dfrac{\log\left(|\log(x_n)|\right)}{n}$.

Problem B1 (Putnam 1960) & A1 (Putnam 1961)

Define

$$A := \{(x,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \,:\, x^y = y^x\}$$ [1]

and find:

$$A, \quad A \cap (\mathbb{Q} \times \mathbb{Q}) \quad \text{and} \quad A \cap (\mathbb{Z} \times \mathbb{Z})\,.$$

Bonus: Find $A \cap \left(\mathbb{Q}^{\text{alg}} \times \mathbb{Q}^{\text{alg}}\right)$ and $A \cap \left(\mathcal{O}_{\mathbb{Q}^{\text{alg}}}(\mathbb{Z}) \times \mathcal{O}_{\mathbb{Q}^{\text{alg}}}(\mathbb{Z})\right)$.

[1] Recall that the exponentiation $a^b := \exp(b \cdot \log(a))$ where $a > 0$ and $b \in \mathbb{R}$ can be extended to $a = 0$ and $b \geq 0$ by:

$$0^b = \begin{cases} 0 & (b > 0), \\ 1 & (b = 0). \end{cases}$$

and extended in the obvious way for $a \in \mathbb{R}^\times$ with $b \in \mathbb{Z}$.