Problem Selection from Past Spring 2025

ETHZ Math Olympiad Club

Fall 2025

Problems: Simon's Favorite Factoring Trick

0.1 AMC 12 (2012)

How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

0.2 AIME (1998)

An $m \times n \times p$ rectangular box has half the volume of an $(m+2) \times (n+2) \times (p+2)$ rectangular box, where m, n, and p are integers, and $m \leq n \leq p$. What is the largest possible value of p?

0.3 BMO (2005)

The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$

Prove that N is a perfect square.

0.4 JBMO (2003)

Let n be a positive integer. A number A consists of $2n$ digits, each of which is 4; and a number B consists of n digits, each of which is 8. Prove that $A + 2B + 4$ is a perfect square.

0.5 AIME (2000)

The system of equations

$$\begin{aligned}
\log_{10}(2000xy) - \log_{10}(x)\log_{10}(y) &= 4, \\
\log_{10}(2yz) - \log_{10}(y)\log_{10}(z) &= 1, \\
\log_{10}(zx) - \log_{10}(z)\log_{10}(x) &= 0,
\end{aligned}$$

has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$.

For these set of problem you may find the follwing trick useful: SFFT is often used in a Diophantine equation where factoring is needed. We let R be any unitary ring. If we have a multipolynomial in two formal variables X and Y, $P(X, Y) \in R[X, Y]$ of the form

$$P(X, Y) = aXY + bX + cY + d \in R[X, Y]$$

with $a \in R^{\times}$. According to *Simon's Favorite Factoring Trick*, this multipolynomial is:

$$P(X,Y) = a\left(X + a^{-1}c\right)\left(Y + a^{-1}b\right) + d - ca^{-1}b.$$

Problem (Wu-Riddles)

Let $n \in \mathbb{N}_{>0}$ and $A, B \in \mathbb{R}^{n\times n}$ be real matrices. Now suppose $I_n - BA$ is invertible, where I_n is the identity matrix. Prove that $I_n - AB$ is also invertible.

Problem (unknown)

An enemy submarine is hidden somewhere along the infinite line $\mathbb{R}$. It travels silently, and you know that its path is described by a fixed rational polynomial rule $\sum_{i=0}^{n} c_i T^i \in \mathbb{Q}[X]$; for each time $t \in \mathbb{N}$, its position is given by

$$x(t) = \sum_{i=0}^{n} c_i t^i \in \mathbb{Q}.$$

However, you do not know $n \in \mathbb{N}$, nor the rational coefficients $(c_i)_{i\in n} \in \mathbb{Q}^n$ that form this rational polynomial rule.

Each unit of time (starting from 0), you are allowed to launch a single torpedo at any chosen rational position. If the submarine is at that position at that time, it is struck and sinks. Assume you possess a potentially countably infinite arsenal of torpedoes and potentially countably infinite time.

Devise a strategy — a sequence of torpedo launches — such that, regardless of the submarine's unknown position, you will **eventually** hit it.

Problem (X-ENS 11 Orals)

Let $(G, \cdot, e_G)$ be a finite non-commutative group. Show that the probability that two elements $x, y \in G$ chosen uniformly at random commute is less than or equal to $\frac{5}{8}$, and show that this bound is tight, i.e., provide an example where the bound is attained.

Problems (Sudakov & Milojevic)

0.6

Let C_1, C_2, C_3 be disjoint circles in the plane of different radii, and let T_{ij} be the intersection point of the common tangent to C_i and C_j for all $1 \leq i < j \leq 3$. Show that the points T_{12}, T_{23}, and T_{13} lie on a common line.

0.7

Several spherical planets, each of radius R, are placed in a greenhouse. On each planet, Mark colors in black the regions that are not visible from any other planet by a single straight-line segment. Prove that the total area of the colored regions, summed over all planets, is exactly $4\pi R^2$.

0.8

There is a pile of silver coins on a table. John holds two pieces of paper and performs the following process: at each step, he can add one gold coin to the table and write the current number of silver coins on one piece of paper, or remove one silver coin from the table and write down the current number of gold coins on the other piece of paper. This process runs until no more silver coins remain on the table. Show that at the end of the process, the sums of the numbers on both pieces of paper are equal.

Problem B6 (Putnam 1985)

Let $n \geq 1$ and $G \leq \mathrm{GL}_n(\mathbb{R})$ be a finite group consisting of real $n \times n$ matrices under matrix multiplication. Suppose the sum of the traces of all elements in G is zero:

$$\sum_{M \in G} \operatorname{tr}(M) = 0.$$

Prove that the sum of the elements of G is the zero matrix:

$$\sum_{M \in G} M = \mathbf{0}_{n \times n}.$$