Problem Set 3

ETHZ Math Olympiad Club

Fall 2025

Problem: unknown

Let $n \geq 2$ be an integer. Choose uniformly $X_1, \ldots, X_n$ from $[0, 1]$. Let p_n be the probability that $X_i + X_{i+1} \leq 1$ for all $i = 1, \ldots, n-1$. Prove that $\lim_{n\to\infty} p_n^{1/n}$ exists and compute it.

Problem: B-3 Putnam 1984

Let $n \geq 1$ be an integer. Assume C and D are chosen at random from $\{1, \ldots, n\}$. Let p_n be the probability that $C + D$ is a perfect square. Compute $\lim_{n\to\infty}(\sqrt{n} \cdot p_n)$. Express the result in the form $(a\sqrt{b} + c)/d$, where a, b, c, d are integers.

Problem: unknown

You're given a machine to which you can input an integer and it outputs an integer. You know the machine just plugs your number into a polynomial and outputs the result, but you don't know which polynomial. You do know that the coefficients are all non-negative integers. The machine is at low battery, and only has energy left for two inputs. How can you guess the polynomial? **Bonus:** Alternatively, you can switch the machine to real number mode, where you can input a real number. This takes twice the battery. Can you guess the polynomial this way?

Problem: 2 Bernoulli Competition 2025

Let $n \geq 1$ be an integer and $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be two $n \times n$-matrices with $\mathrm{rank}(A) = \mathrm{rank}(B)$. Assume there exists an integer $k \geq 1$ such that

$$A^{k+1}B^k = A.$$

Show that then also

$$B^{k+1}A^k = B.$$

Problem: 11446, The American Mathematical Monthly Vol. 116, No. 7

Prove or disprove: there exist 2×2 symmetric integer matrices A and B such that no element of the multiplicative semigroup generated by A and B can be written in two different ways. (Thus, $A, B, AA, AB, BA, BB, AAA, AAB, \ldots$ are all different.)