

# Problem Set Week 10

Math Olympiad Club Zurich

Spring 2025

## Problem B3 Putnam 2001

For each  $n \in \mathbb{N}$ , let  $\langle n \rangle$  denote the closest integer to  $\sqrt{n}$ . Find  $\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$ .

## Problem 3 Bernoulli Competition 2024

Suppose  $x_0 \in \mathbb{R}$  and  $x_{n+1} = \sum_{i=0}^n (-1)^i \sin(x_i)$  for  $n \geq 1$ .

1) What is the range for the  $x_0$  such that  $\lim_{n \rightarrow \infty} x_n$  exists? What is the value of the limit depending on  $x_0$  in the range?

2) Suppose  $x_0 = 1/4$ . Find  $\lim_{n \rightarrow \infty} \frac{\log(|\log(x_n)|)}{n}$ .

## Problem unknown

Let  $(S, \cdot)$  be a non-empty magma; that is  $S$  be a non-empty set, with an internal binary operation. Suppose that it satisfies the following:

- $(S, \cdot)$  forms a semi-group i.e the binary operation  $\cdot$  is associative. -The binary operation  $\cdot$  is injective in left and right coordinate  $\forall a, b, c \in S$ ;

$$\cdot((a, b)) = \cdot((a, c)) \Rightarrow b = c \text{ i.e. } a \cdot b = a \cdot c \Rightarrow b = c$$

$$\cdot((b, a)) = \cdot((c, a)) \Rightarrow b = c \text{ i.e. } b \cdot a = c \cdot a \Rightarrow b = c$$

$\forall a \in S, a^{\mathbb{Z}_{\geq 1}} := \{a^n \mid n \in \mathbb{Z}_{\geq 1}\}$  is finite, where  $a^1 := a$  and for  $n > 1$ ,  $a^n = a \cdot a^{n-1}$ . (Note that this definition could be independent where we start to compute since  $\cdot$  is associative.)

Show that  $S$  can be turned into a group where the group operation is  $\cdot$ .

## Problem unknown

There is an odd number  $(2 \cdot n + 1 > 0)$  of stones with real weights satisfying the following property: if we remove any stone from the  $2 \cdot n + 1$ , then there is a way to partition the rest of the stones into two sets of size  $n$ , such that the sum of the weights of the stones in both sets is equal. Show that all stones have the same weight.

## Problem B-6 IMC 2024

Show that any function  $f : \mathbb{Q} \rightarrow \mathbb{Z}$  satisfy the following propertie:

$$\exists a, b, c \in \mathbb{Q} \text{ with } a < b < c \text{ such that } f(a), f(c) \leq f(b)$$