Problem Set Week 11

Math Olympiad Club Zurich

Spring 2025

Problem 1 (Bernoulli Competition 2023)

1. Let $A = \{1, 2, \ldots, 100\}$ be the set of integers between 1 and 100.

(a) Let $B \subset A$ be a subset that doesn't contain two consecutive integers. What is the maximal cardinality of B?

(b) Let $C \subset A$ be a subset such that there is no n for which n and $2n$ are both in C. What is the maximal cardinality of C?

Problem (Selected Real Analysis Problem)

For each function $g \in \{-id_{\mathbb{R}}, \exp, x \mapsto x^2 - 2\}$, determine wheter there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ f = g$.

Bonus: Solve the same problem with $g \in \{\cos, \sin\}$.

Problem B4 (Putnam 2001)

Let $S := \mathbb{Q} \setminus \{-1, 0 - 1\}$. Define $f : S \to S$ by $f(x) = x - \frac{1}{x}$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

Problem (Hongler)

We have a linked list of elements, $x_0 \to x_1 \to \ldots \to x_n$, where $n \geq 1$ is unknown (but it is known that the list is finite). When at x_0, we have a pointer to go to x_1, which leads to x_2, and so on, until we reach x_n, where we learn that it is the end. We have a bounded amount of memory (at least we can stock one elements x_i) (we cannot simply store the entire list in an array and choose an element from the array once we reach the end).

(a) How can we select a random element in the linked list $x_1 \to \ldots \to x_n$, uniformly, if we are allowed to traverse the list only once?

(b) Why might solving this problem be useful in practice?

Problem (Hongler)

Let $U \subset \mathbb{C}$ be a domain containing the disc $\mathbb{D}$; that is, $\mathbb{D} \subset U$, and let $f : U \to \mathbb{C}$ be a holomorphic function. Show that if $f(\partial\mathbb{D}) = \gamma$ is a simple loop (i.e., it can be parametrised by a continuous path which intersects only at the endpoints), and if $f|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \gamma$ is injective, then $f|_{\mathbb{D}}$ is injective.