

# Problem Set Wee 2 Solutions

Math Olympiad Club Zurich

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## Problem: 1 Pan African 2018

Find all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that

$$(f(x+y))^2 = f(x^2) + f(y^2)$$

for all  $x, y \in \mathbb{Z}$ .

### Solution:

Plug in  $y = -x$ , and let  $c = f(0)$ . We have  $f(x^2) = \frac{c^2}{2}$ . Plugging in  $y = 0$ , we have  $f(x)^2 = \frac{c^2}{2} + c$ . Plugging in  $x = 0$  in this gives  $c^2 = \frac{c^2}{2} + c$   $c = 0$  or  $c = 2$ . Now if  $c = 0$ ,  $f \equiv 0$  is the only solution. Else we have  $f(x) = \pm 2$  for all  $x$ . This means  $f(x^2) + f(y^2) = 4$ , always, so for all  $x$  being perfect squares,  $f(x) = 2$ . So the solution in this case becomes  $f(x) = 2$  for all perfect square  $x$  and  $\pm 2$  for all other  $x$ . It is easy to check that the solutions mentioned above work. Thus:

$$\begin{aligned} & \left\{ f : \mathbb{Z} \rightarrow \mathbb{Z} \mid \forall x, y \in \mathbb{Z} (f(x+y))^2 = f(x^2) + f(y^2) \right\} \\ &= \{0\} \cup \left\{ 2\mathbb{1}_A - 2\mathbb{1}_{\mathbb{Z} \setminus A} \mid \exists A \subset \mathbb{Z}, \square_{\mathbb{N}} \subset A \right\} \end{aligned}$$

where  $\square_{\mathbb{N}} := \{n^2 \mid n \in \mathbb{N}\}$ .

## Problem: B-2 IMC 2012

Define the sequence  $(a_n)_{n \geq 0}$  inductively by  $a_0 = 1$ ,  $a_1 = \frac{1}{2}$ , and

$$a_{n+1} = \frac{na_n^2}{1 + (n+1)a_n} \quad \text{for } n \geq 1.$$

Show that the series

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$$

converges and determine its value.

### Solution:

By induction, we establish that  $a_i > 0$  for all  $i \geq 0$ , ensuring that the partial sums are well defined and form an increasing sequence. Furthermore, we observe that

$$ka_k = \frac{(1 + (k+1)a_k)a_{k+1}}{a_k} = \frac{a_{k+1}}{a_k} + (k+1)a_{k+1} \quad \text{for all } k \geq 1.$$

Summing from  $k = 0$  to  $n$ , we obtain

$$\sum_{k=0}^n \frac{a_{k+1}}{a_k} = \frac{a_1}{a_0} + \sum_{k=1}^n (ka_k - (k+1)a_{k+1}) = \frac{1}{2} + 1 \cdot a_1 - (n+1)a_{n+1} = 1 - (n+1)a_{n+1}$$

for all  $n \geq 1$ . A quick verification confirms that this equality also holds for  $n = 0$ .

Since  $(n+1)a_{n+1} > 0$ , we deduce that for each  $n \geq 0$ ,

$$\sum_{k=0}^n \frac{a_{k+1}}{a_k} = 1 - (n+1)a_{n+1} < 1.$$

This implies that the partial sums are bounded, and thus, the series  $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$  is convergent (converging to the supremum of the partial sums). Consequently, the sequence  $\frac{a_{k+1}}{a_k}$  must tend to zero. In particular, there exists an index  $N \in \mathbb{N}$  such that  $\frac{a_{n+1}}{a_n} < \frac{1}{2}$  for all  $n > N$ .

For all  $n > N$ , we then have

$$a_n = \prod_{i=N}^{n-1} \frac{a_{i+1}}{a_i} < \frac{1}{a_N 2^{n-1-N}} = \frac{a_N^{-1} 2^N + 1}{2^n}.$$

In particular, for all  $n > N$ ,

$$na_n < (a_N^{-1} 2^N + 1) \left( \frac{n}{2^n} \right).$$

Since  $\frac{n}{2^n} \rightarrow 0$  as  $n \rightarrow +\infty$ , it follows that  $na_n \rightarrow 0$  as  $n \rightarrow +\infty$ . Therefore,

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{a_{k+1}}{a_k} = \lim_{n \rightarrow \infty} (1 - (n+1)a_{n+1}) = 1.$$

## Problem: 5 Pan African 2018

Let  $a, b, c$  and  $d$  be non-zero pairwise different real numbers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4 \text{ and } ac = bd.$$

Show that

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} \leq -12$$

and that  $-12$  is the maximum.

### Solution:

First write  $d = \frac{ac}{b}$  so we have  $\frac{a}{b} + \frac{b}{a} + \frac{c}{b} + \frac{c}{a} = 4$  and we want to show that  $\frac{a}{c} + \frac{c}{a} + \frac{b^2}{ac} + \frac{ac}{b^2}$  or equivalently  $(\frac{a}{b} + \frac{b}{a})(\frac{c}{b} + \frac{b}{c}) \leq -12$ .

Set  $\frac{a}{b} + \frac{b}{a} = s$  and  $\frac{c}{b} + \frac{b}{c} = t$ . We have  $s + t = 4$  and we want  $st \leq -12$ .

We have  $t = 4 - s$  and so we need  $s(4 - s) \leq -12 \Rightarrow s^2 - 4s - 12 \geq 0$ . If  $s \leq -2$ , this holds (indeed, set  $s = -2 + k$  and we have  $s^2 - 4s - 12 = (k - 2)^2 - 4(k - 2) - 12 = k^2 - 8k - 16 = (k - 4)^2 \geq 0$ ).

So assume that  $s \geq -2$ . Similarly, assume  $t \geq -2$ . We have then  $\frac{a}{b} + \frac{b}{a} \geq -2 \Rightarrow \frac{(a+b)^2}{ab} > 0$ , so  $ab > 0$  and  $bc > 0$ , so  $a, b, c$ , are of the same sign, so  $4 = \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} \geq 2 + 2 = 4$ , so  $a = b = c$ , contradiction, since  $a \neq b \neq c \neq a$ .

So, at least one of  $s, t$  is  $\leq -2$  and the stated inequality holds.

The equality happens e.g. when  $(a, b, c, d) = (2\sqrt{2} - 3, 3 - 2\sqrt{2}, 1, -1)$  (and this shows that  $-12$  is the maximum).

## Problem: 3 Silk Road 2019

Find all pairs  $(a, n)$  of positive natural numbers such that  $\varphi(a^n + n) = 2^n$ .

( $\varphi(n)$  is the Euler function, that is, the number of integers from 1 up to  $n$ , relatively prime to  $n$ .)

### Solution:

Using the estimation  $\varphi(b) > \sqrt{b}$  (classic) for all  $b > 6$ , we first check the (finitely many) cases when a solution  $(a, n)$  satisfies  $a^n + n \leq 6$  to find that only:

$$(a, n) \in \{(2, 1), (3, 1), (5, 1)\}$$

works.

Now for a solution  $(a, n)$  with  $a^n + n > 6$  we get using the lower bound estimation:

$$\varphi(a^n + n) \stackrel{\text{hyp}}{=} 2^n > \sqrt{a^n + n}.$$

- If  $a \geq 4$  the inequality cannot be true as then:

$$2^n > \sqrt{a^n + n} \geq \sqrt{4^n + n} \geq \sqrt{2^{2n}} = 2^n,$$

so we have to check the cases when  $a = 1, 2, 3$ .

- If  $a = 1$  then using the trivial upper bound estimation  $\varphi(b) \leq b - 1$  we get  $n \geq \varphi(n + 1) = 2^n$ , which never holds for  $n \geq 1$ .
- If  $a = 2$  or  $a = 3$  this is more subtle. To have  $3^n + n > 6$  we need for  $a = 3$  that  $n \geq 2$  and for  $a = 2$  that  $n \geq 3$ . We see that  $(3, 3)$  works: indeed  $\varphi(30) = 30 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) = 8 = 2^3$ . Since  $(3, 2)$  and  $(2, 3)$  don't satisfy the property (easy to check) and  $(3, 3)$  does, we can assume  $n \geq 4$ . Now we claim that there is no integer  $n \geq 4$  satisfying for  $a \in \{2, 3\}$ :

$$\varphi(a^n + n) = 2^n.$$

We proceed as follows:

If an integer  $m \geq 2$  satisfies

$$\varphi(m) = 2^k,$$

then every odd prime  $p$  dividing  $m$  must satisfy

$$p - 1 = 2^j.$$

Indeed, writing  $m = 2^k \cdot p_1^{e_1} \cdots p_r^{e_r}$  for some distinct odd primes  $p_i$ ,  $k \geq 0$  and  $e_1, \dots, e_r \geq 1$ , since  $\varphi(m) = 2^{k-1} \cdot p_1^{e_1-1} \cdots p_r^{e_r-1} (p_1 - 1) \cdots (p_r - 1)$ , the only way for  $\varphi(m)$  to be a power of 2 is that:

- For each  $i \in \{1, \dots, r\}$ , we have  $e_i = 1$ .
- Every odd prime  $p_i$  satisfies  $p_i - 1 = 2^{a_i}$  for some  $a_i \geq 0$ . In other words, every odd prime factor of  $m$  is a Fermat prime<sup>1</sup>.

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<sup>1</sup>A Fermat prime is a prime number of the form  $2^j + 1$ . One can easily see (mental exercise) that  $j$  is necessarily **again** a power of 2. Indeed, if  $j$  is not a power of 2, then there is an odd prime  $p \mid j$  and  $2^j + 1 = \left(2^{\frac{j}{p}}\right)^p - (-1)^p = \left(2^{\frac{j}{p}} + 1\right) \left(\sum_{s=0}^{p-1} (-1)^{p-1-s} \left(2^{\frac{j}{p}}\right)^s\right)$ , which is a non-trivial factorization of  $2^j + 1$  since  $1 < 2^{\frac{j}{p}} + 1 < 2^j + 1$ , contradicting the fact that  $2^j + 1$  is a prime number. *Nota bene*: the only known Fermat primes are  $3 = 2^{(2^0)} + 1$ ,  $5 = 2^{(2^1)} + 1$ ,  $17 = 2^{(2^2)} + 1$ ,  $257 = 2^{(2^3)} + 1$ , and  $65537 = 2^{(2^4)} + 1$ , and we don't know if there are infinitely many of them.

Hence  $2^n = \varphi(m) = 2^{k-1} \prod_{i=1}^r 2^{a_i}$ , and so the following equality holds:

$$k - 1 + \sum_{i=1}^r a_i = n.$$

If we let  $n \geq 4$ ,  $a \in \{2, 3\}$ , and

$$m = a^n + n,$$

with  $\varphi(m) = 2^n$ , then with the same notation as before:

$$a^n + n = 2^k \cdot p_1 \cdots p_r = 2^k (2^{a_1} + 1) \cdots (2^{a_r} + 1),$$

for  $k \geq 0$  and some distinct odd primes  $p_i = 2^{a_i} + 1$  with  $a_i \geq 1$  (as  $p_i$  is odd) (the  $a_i$  are powers of 2, but this is not needed here). Moreover, the equality  $k - 1 + \sum_{i=1}^r a_i = n$  holds. Without loss of generality, order the exponents:

$$1 \leq a_1 < \cdots < a_r.$$

Now let us attack each case ( $a = 2$  or  $a = 3$  with  $n \geq 4$ ) separately:

-If  $a = 2$ , then if  $6 < 2^n + n$  is a prime number, this means that  $k = 0$  and  $r = 1$ , and  $-1 + a_1 = n$ , so that  $2^n + n = 2^{a_1} + 1 = 2^{n+1} + 1$ , and thus  $n = 2^n + 1 > 2^n > n$ , which is a contradiction. Thus,  $2^n + n$  satisfying the property cannot be a prime number. Hence, it is composite. We have the following classical upper bound for the Euler totient for a composite natural number  $m$ :

$$\varphi(m) = m \prod_{p|m} \left(1 - \frac{1}{p}\right) \leq m \left(1 - \frac{1}{\min\{p \in \mathbb{P} \mid p \mid m\}}\right) = m - \frac{m}{\min\{p \in \mathbb{P} \mid p \mid m\}}.$$

And since  $m$  is composite (we use it here),  $\min\{p \in \mathbb{P} \mid p \mid m\} \leq \sqrt{m}$ , so that:

$$\varphi(m) \leq m - \sqrt{m}.$$

Using this upper bound estimation, we get:

$$2^n + n - \sqrt{2^n + n} \geq \varphi(2^n + n) = 2^n.$$

Noticing that the function  $f$  defined on  $\mathbb{R}_+$  by  $f(x) = x - \sqrt{2^x + x}$  is always negative (because for all  $x \geq 0$  we have  $x(x-1) < x^2 < 2^x$  since  $\ln(x) < \frac{x}{2}$ ), we conclude that the above inequality cannot hold for  $n \geq 1$ . This shows that there is no integer  $n \geq 4$  such that  $\varphi(2^n + n) = 2^n$ .

-If  $a = 3$ , then by induction, one has the following fact:

$$t \geq 1 \implies 2^t + 1 \leq 3^t \text{ with equality only at } t = 1,$$

$$t \geq 3 \implies 2^t < 2^t + 1 < 3^{t-1}.$$

With this in mind, we obtain easily:

If there exists  $a_i$  with  $a_i \geq 3$  (in particular, this occurs if  $r \geq 3$ , as then for  $r \geq i > 2$  we have  $a_i \geq 3$ ), then  $2^{a_i} + 1 < 3^{a_i-1}$ , which yields:

$$3^n < 3^n + n = 2^k (2^{a_1} + 1) \cdots (2^{a_r} + 1) < 2^k 3^{a_i-1} \prod_{\substack{t=1 \\ t \neq i}}^r 3^{a_t} = 2^k 3^{n-k} \leq 3^n,$$

which is a contradiction. Thus, in particular,  $1 \leq r \leq 2$ , and all  $a_i$  belong to  $\{1, 2\}$ .

If  $k \geq 3$ , then:

$$3^n < 3^n + n = 2^k (2^{a_1} + 1) \cdots (2^{a_r} + 1) \leq 2^k 3^{a_1} \cdots 3^{a_r} \leq 2^k 3^{n+1-k} \leq 3^{k-1} 3^{n+1-k} = 3^n,$$

which is again a contradiction, so  $0 \leq k \leq 2$ .

Summarizing, we have that a solution  $3^n + n$  must be in the following set:

$$\begin{aligned} & \left\{ 3^j + j \mid j \in \mathbb{N}, j \geq 4 \right\} \cap \left\{ 2^k (2^a + 1)^{\epsilon_a} (2^b + 1)^{\epsilon_b} \mid k \in \{0, 1, 2\}, a, b \in \{1, 2\}, \epsilon_a, \epsilon_b \in \{0, 1\}, a \neq b \right\} \\ &= \left\{ 3^j + j \mid j \in \mathbb{N}, j \geq 4 \right\} \cap \left\{ 2^k 3^{\epsilon} 5^{\epsilon'} \mid k \in \{0, 1, 2\}, \epsilon, \epsilon' \in \{0, 1\} \right\} \\ &= \{85\} \cap \{1, 3, 5, 9, 15, 25, 2, 6, 10, 18, 30, 50, 4, 12, 20, 36, 60, 100\} = \emptyset. \end{aligned}$$

This shows that there is no integer  $n \geq 4$  such that  $\varphi(3^n + n) = 2^n$ .

To summarize, the only pairs that work are  $\{(2, 1), (3, 1), (3, 3), (5, 1)\}$ , and this concludes.