# Problem Set Week 3

ETHZ Math Olympiad Club

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## Problem in example page 140 (PUTNAM and BEYOND)

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function, with positive second derivative. Prove that

$$f\left(x + f'(x)\right) \geq f(x),$$

for any real number $x$.

## Problem A-2 (IMC 2011)

Does there exist a real $3 \times 3$ matrix $A$ such that

$$\operatorname{tr}(A) = 0 \quad \text{and} \quad A^2 + A^T = I_3,$$

where $\operatorname{tr}(A)$ denotes the trace of $A$, $A^T$ is the transpose of $A$, and $I_3$ is the $3 \times 3$ identity matrix?

## Problem B-2 (IMC 2014)

Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \leq i < j \leq n} a_{ii} a_{jj} \geq \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.

## Problem 414 (PUTNAM and BEYOND)

For any real number $\lambda \geq 1$, denote by $f(\lambda)$ the real solution to the equation

$$x(1 + \ln x) = \lambda.$$

Prove that

$$\lim_{\lambda \to \infty} \frac{f(\lambda)}{\frac{\lambda}{\ln \lambda}} = 1.$$

## Problem A-4 (IMC 2014)

Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with

$$1 < p_1 < \ldots < p_k.$$

Prove that $e_1$ is an even number.
A number $n$ is *perfect* if $s(n) = 2n$, where $s(n) = \sum_{\mathbb{N} \ni d | \mathbb{Z} n} d$ is the sum of the divisors of $n$.