Problem Set Week 3 Solutions

ETHZ Math Olympiad Club

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Problem in example page 140 (PUTNAM and BEYOND)

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function, with positive second derivative. Prove that

$$f(x + f'(x)) \geq f(x),$$

for any real number x.

Solution:

If x is such that $f'(x) = 0$, then the relation holds with equality. Else $f'(x) \neq 0$ and then the following set has non empty interior:

$$[\, x + f'(x),\, x\,] \sqcup [\, x,\, x + f'(x)\,]$$

It is clear that one of these interval is non empty and the second is empty. The mean value theorem applied on the non empty interval yields the existence of $c \in [\, x + f'(x),\, x\,] \sqcup [\, x,\, x + f'(x)\,]$ such that:

$$f'(c) = \frac{f\left(x + f'(x)\right) - f(x)}{(x + f'(x)) - x} = \frac{f(x + f'(x)) - f(x)}{f'(x)},$$

i.e. $f'(c)f'(x) = f(x + f'(x)) - f(x)$.

If $f'(x) > 0$ then $c \in [\, x,\, x + f'(x)\,]$ and because the second derivative is positive, f' is increasing; hence $0 < f'(x) < f'(c)$. Therefore $f(x + f'(x)) - f(x) > 0$.

If $f'(x) < 0$ then $c \in [\, x + f'(x),\, x\,]$ and because the second derivative is positive, f' is increasing; hence $f'(c) < f'(x) < 0$. Therefore $f(x + f'(x)) - f(x) > 0$.

In all cases:

$$f(x + f'(x)) \geq f(x)$$

Problem A-2 (IMC 2011)

Does there exist a real 3×3 matrix A such that

$$\operatorname{tr}(A) = 0 \quad \text{and} \quad A^2 + A^T = I_3,$$

where $\operatorname{tr}(A)$ denotes the trace of A, A^T is the transpose of A, and I_3 is the 3×3 identity matrix?

Solution:

We claim that no such real 3×3 matrix A exists. Suppose, for contradiction, that a matrix $A \in \mathbb{R}^{3\times 3}$ exists with $\operatorname{tr}(A) = 0$ and $A^2 + A^T = I_3$. Taking the transpose of the second equation, we obtain

$$I_3 = I_3^T = \left(A^2 + A^T\right)^T = \left(A^2\right)^T + A = \left(A^T\right)^2 + A.$$

Using the original assumption $A^2 + A^T = I_3$, we substitute:

$$I_3 = \left(I_3 - A^2\right)^2 + A = I_3 - A^2 - A^2 + A^4 + A = A^4 - 2A^2 + A + I_3.$$

Thus, we obtain the matrix polynomial equation

$$P(A) = 0_3 \quad \text{with} \quad P(X) := X^4 - 2X^2 + X \in \mathbb{R}[X],$$

where 0_3 denotes the 3×3 zero matrix. We now factor the polynomial:

$$P(X) = X^4 - 2X^2 + X = X(X-1)\left(X^2 - X - 1\right).$$

It follows that the minimal polynomial of A must divide $P(X)$, and hence the eigenvalues of A must be among

$$\left\{0, 1, \tfrac{-1 \pm \sqrt{5}}{2}\right\}.$$

Recalling that the trace of a matrix is the sum of its eigenvalues (counted with multiplicity), we obtain:

$$0 = \operatorname{tr}(A) = \sum_{\lambda \in \sigma(A)} \dim_{\mathbb{R}}(\ker(A - \lambda I_3)) \cdot \lambda$$

where $\sigma(A)$ denotes the spectrum of A. Moreover, taking the trace of both sides of $A^2 + A^T = I_3$ gives:

$$3 = 3 - 0 = \operatorname{tr}(I_3) - \operatorname{tr}(A) = \operatorname{tr}\left(I_3 - A^T\right) = \operatorname{tr}\left(A^2\right).$$

Thus,

$$\operatorname{tr}(A^2) = \sum_{\mu \in \sigma(A^2)} \dim_{\mathbb{R}}\left(\ker\left(A^2 - \mu I_3\right)\right) \cdot \mu.$$

Since $\sigma\left(A^2\right) = \{\lambda^2 \mid \lambda \in \sigma(A)\}$[1], we obtain:

$$3 = \sum_{\lambda \in \sigma(A)} \dim_{\mathbb{R}} \left(\ker\left(A^2 - \lambda^2 I_3\right)\right) \lambda^2.$$

By direct case-checking of the eigenvalues $\sigma(A) \subset \left\{0, 1, \frac{-1 \pm \sqrt{5}}{2}\right\}$, one easily verifies that the two conditions on $\operatorname{tr}(A)$ and $\operatorname{tr}\left(A^2\right)$ cannot be simultaneously satisfied. This yields a contradiction.

Hence, no real 3×3 matrix A satisfies both $\operatorname{tr}(A) = 0$ and $A^2 + A^T = I_3$.

[1] In general, for any $n \times n$ matrix B over $\mathbb{C}$ (or $\mathbb{R}$), write B in its Jordan canonical form

$$B = VJV^{-1},$$

where J is a block-diagonal matrix consisting of Jordan blocks corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$ of B, and V is invertible. Applying a polynomial $f(X) \in \mathbb{C}[X]$ to B gives:

$$f(B) = f(VJV^{-1}) = Vf(J)V^{-1}.$$

By a simple computation, the matrix $f(J)$ is a block-diagonal matrix consisting of Jordan blocks corresponding to the diagonal entries $f(\lambda_1), \ldots, f(\lambda_k)$. By uniqueness of the Jordan form (up to block permutation), $f(J)$ is a block-diagonal matrix consisting of Jordan blocks corresponding to the eigenvalues of $f(B)$ showing that the eigenvalues of $f(B)$ are precisely $\{f(\lambda) \mid \lambda \in \sigma(B)\}$, although the eigenvectors need not coincide.

Problem B-2 (IMC 2014)

Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1\le i<j\le n} a_{ii}a_{jj} \ge \sum_{1\le i<j\le n} \lambda_i\lambda_j,$$

and determine all matrices for which equality holds.

Solution:

Eigenvalues of a real symmetric matrix are real, hence the inequality is well-defined. The trace of a matrix equals the sum of its eigenvalues. For matrix A,

$$\sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i.$$

Squaring both sides, we obtain:

$$\left(\sum_{i=1}^n a_{ii}\right)^2 = \left(\sum_{i=1}^n \lambda_i\right)^2.$$

Expanding both sides gives:

$$\sum_{i=1}^n a_{ii}^2 + 2\sum_{1\le i<j\le n} a_{ii}a_{jj} = \sum_{i=1}^n \lambda_i^2 + 2\sum_{1\le i<j\le n} \lambda_i\lambda_j.$$

It therefore suffices to show the inequality:

$$\sum_{i=1}^n a_{ii}^2 \le \sum_{i=1}^n \lambda_i^2.$$

The matrix A^2, which equals A^TA for symmetric A, has eigenvalues $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ for the same reason as the previous problem. The trace of A^TA is the square of the Frobenius norm of A:

$$\operatorname{tr}\left(A^TA\right) = \sum_{i,j=1}^n a_{ij}^2 = \operatorname{tr}\left(A^2\right) = \sum_{i=1}^n \lambda_i^2.$$

Obviously $\sum_{i=1}^n a_{ii}^2 \le \sum_{i,j=1}^n a_{ij}^2$, the inequality $\sum_{i=1}^n a_{ii}^2 \le \sum_{i=1}^n \lambda_i^2$ follows. One sees then that equality holds if and only if $\sum_{i=1}^n a_{ii}^2 = \sum_{i,j=1}^n a_{ij}^2$ that is if and only if all off-diagonal entries of A are zero, i.e., A is diagonal.

Remark. The same result holds for Hermitian matrices as for Hermitian matrices, diagonal entries and eigenvalues are also real.

Problem 414 (PUTNAM and BEYOND)

For any real number $\lambda \geq 1$, denote by $f(\lambda)$ the real solution to the equation

$$x(1+\ln x) = \lambda.$$

Prove that

$$\lim_{\lambda \to +\infty} \frac{f(\lambda)}{\frac{\lambda}{\ln \lambda}} = 1.$$

Solution:

The function $h : [1, +\infty[\to [1, +\infty[$ given by $h(t) = t(1 + \ln t)$ is strictly increasing, and $h(1) = 1$, $\lim_{t\to+\infty} h(t) = +\infty$. Hence h is bijective, and its inverse is clearly the function $f : [1, \infty) \to [1, \infty)$, $\lambda \to f(\lambda)$ satisfying $\lambda = f(\lambda)(1 + ln(f(\lambda)))$. Since h is differentiable with $h'(t) = 2 + ln(t)$ which never vanishes for $t \in [1; +\infty[$ so is f, and

$$f'(\lambda) = \frac{1}{h'(f(\lambda))} = \frac{1}{2 + \ln f(\lambda)}.$$

Also, since h is strictly increasing and $\lim_{t\to+\infty} h(t) = +\infty$, $f(\lambda)$ is strictly increasing, and its limit at + infinity is also + infinity. Using the defining relation for $f(\lambda)$, we see that for $\lambda \geq 1$:

$$\frac{f(\lambda)}{\frac{\lambda}{\ln \lambda}} = \ln \lambda \cdot \frac{f(\lambda)}{\lambda} = \frac{\ln \lambda}{1 + \ln f(\lambda)}.$$

Now we apply L'Hôpital's theorem and obtain

$$\lim_{\lambda\to+\infty} \frac{f(\lambda)}{\frac{\lambda}{\ln \lambda}} = \lim_{\lambda\to+\infty} \frac{\frac{1}{\lambda}}{\frac{1}{f(\lambda)} \cdot \frac{1}{2+\ln f(\lambda)}} = \lim_{\lambda\to+\infty} \frac{f(\lambda)}{\lambda}(2 + \ln f(\lambda))$$

$$= \lim_{\lambda\to+\infty} \frac{2 + \ln f(\lambda)}{1 + \ln f(\lambda)} = 1 + \lim_{\lambda\to+\infty} \frac{1}{1 + \ln f(\lambda)} = 1,$$

where the last equality follows from the fact $\lim_{\lambda\to+\infty} \ln f(\lambda) = +\infty$. Therefore, the required limit is equal to 1.

Problem A-4 (IMC 2014)

Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with

$$1 < p_1 < \ldots < p_k.$$

Prove that e_1 is an even number.
A number n is *perfect* if $s(n) = 2n$, where $s(n) = \sum_{\mathbb{N} \ni d |_{\mathbb{Z}} n} d$ is the sum of the divisors of n.

Solution:

Suppose that e_1 is odd, contrary to the statement. We know that

$$s(n) = \prod_{i=1}^{k} \left(\sum_{j=0}^{e_i} p_i^j \right) = 2n = 2p_1^{e_1} \cdots p_k^{e_k}.$$

Since e_1 is an odd number, $p_1 + 1$ divides the first factor[2]:

$$\sum_{j=0}^{e_1} p_1^j = \left(\sum_{j=0}^{\frac{e_1-1}{2}} p_1^{2j} \right) + \left(\sum_{j=0}^{\frac{e_1-1}{2}} p_1^{2j+1} \right) = (1 + p_1) \left(\sum_{j=0}^{\frac{e_1-1}{2}} p_1^{2j} \right)$$

so $p_1 + 1$ divides $2n$. Due to $p_1 + 1 > 2$, at least one of the primes $p_1, \ldots, p_k$ divides $p_1 + 1$. The primes $p_3, \ldots, p_k$ are greater than $p_1 + 1$ and p_1 cannot divide $p_1 + 1$, clearly p_1 doesn't divide $p_1 + 1$ (else p_1 would divide 1) so p_2 must divide $p_1 + 1$ i.e. $\exists t \in \mathbb{N}^*$ with $p_1 + 1 = tp_2$. Since $tp_2 = p_1 + 1 < 2p_1 < 2p_2$, this is possible only if $t = 1$ i.e. $p_2 = p_1 + 1$, therefore $p_1 = 2$ and $p_2 = 3$ (the only two consecutive primes are 2 and 3). Hence, $6 \mid n$.

Now $n, \frac{n}{2}, \frac{n}{3}, \frac{n}{6}$ and 1 are distinct divisors of n, so by definition

$$2n \overset{n \text{ is perfect}}{=} s(n) \geq n + \frac{n}{2} + \frac{n}{3} + \frac{n}{6} + 1 = n \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \right) + 1 = 2n + 1 > 2n,$$

which is a contradiction.

Remark. The perfect numbers with a first power odd must therefore be smaller than 6. One can check that the only perfect number $n \leq 6$ is $6 = 2 \cdot 3$ and this one has an odd first power.

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[2] one could also see it as follows. Since:

$$\left\{ (1 + p_1) \left(\sum_{j=0}^{\frac{e_1-1}{2}} (-p_1)^j \right), (1 - p_1) \left(\sum_{j=0}^{\frac{e_1-1}{2}} (-p_1)^j \right) \right\} = \left\{ \left(1 - p_1^{\frac{e_1+1}{2}} \right), \left(1 + p_1^{\frac{e_1+1}{2}} \right) \right\}$$

where identifying which one is which one depends on the parity of $\frac{e_1-1}{2}$. We must have that

$$\sum_{j=0}^{e_1} p_1^j = \frac{1 - p_1^{e_1+1}}{1 - p_1} = \frac{\left(1 - p_1^{\frac{e_1+1}{2}} \right) \left(1 + p_1^{\frac{e_1+1}{2}} \right)}{1 - p_1} = \frac{(1 + p_1) \left(\sum_{j=0}^{\frac{e_1-1}{2}} (-p_1)^j \right) (1 - p_1) \left(\sum_{j=0}^{\frac{e_1-1}{2}} (-p_1)^j \right)}{1 - p_1}$$

$$= (1 + p_1) \left(\sum_{j=0}^{\frac{e_1-1}{2}} (-p_1)^j \right)^2.$$

Comparing with the previous factorization and using its uniqueness yields:

$$\left(\sum_{j=0}^{\frac{e_1-1}{2}} p_1^{2j} \right) = \left(\sum_{j=0}^{\frac{e_1-1}{2}} (-p_1)^j \right)^2$$