Problem Set Week 4 Solutions

ETHZ Math Olympiad Club

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Problem (unknown)

Find all real solutions to the equation

$$9^x + 4^x + 2^x = 8^x + 6^x + 1.$$

Solution:

It is easy to see that $x = 0$, $x = 1$, and $x = 2$ are solutions. So the equation has at least 3 distinct real solutions. Let us introduce the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 9^x + 4^x + 2^x - 8^x + 6^x + 1.$$

As stated above, f has at least 3 distinct zeros. We claim there are no other roots. By Rolle's theorem, if a function $g : \mathbb{R} \to \mathbb{R}$ has at least $n \geq 2$ zeros $x_1 < \cdots < x_n$, then the function $g'(x)$ has at least $n-1$ zeros $y_1 < \cdots < y_{n-1}$, where for each $i = 1, \ldots, n-1$, we have $x_i < y_i < x_{i+1}$. In particular, since for each $a \in \mathbb{R}_{>0}$, $a^{-x}g(x)$ has at least n zeros, we have that $\left(a^{-x}g(x)\right)'$ has at least $n-1$ zeros, and so does the function

$$h_a g(x) = a^x \left(a^{-x}g(x)\right)' = g'(x) - \ln(a)g(x).$$

Suppose f has another zero not in $\{0, 1, 2\}$. Then f has at least 4 zeros, and thus

$$h_1 f(x) = f'(x) = \ln(9)9^x + \ln(4)4^x + \ln(2)2^x - \ln(8)8^x - \ln(6)6^x$$

has at least 3 zeros, which then implies that

$$h_6 h_1 f(x) = f''(x) - \ln(6)f'(x)$$
$$= \ln\left(\frac{9}{6}\right)\ln(9)9^x + \ln\left(\frac{4}{6}\right)\ln(4)4^x + \ln\left(\frac{2}{6}\right)\ln(2)2^x - \ln\left(\frac{8}{6}\right)\ln(8)8^x$$

has at least 2 zeros, which again implies that

$$h_8 h_6 h_1 f(x) = \ln\left(\frac{9}{8}\right)\ln\left(\frac{9}{6}\right)\ln(9)9^x - \ln\left(\frac{4}{8}\right)\ln\left(\frac{4}{6}\right)\ln(4)4^x - \ln\left(\frac{2}{8}\right)\ln\left(\frac{2}{6}\right)\ln(2)2^x$$

has at least 1 zero.

The function $h_8 h_6 h_1 f(x)$ is of the form $k_2 2^x + k_4 4^x + k_9 9^x$, for $k_2, k_4, k_9 > 0$ and hence is always positive. Therefore, $h_8 h_6 h_1 f(x)$ cannot have any real zero. This is a contradiction to the assumptions hence the solutions of the original equation are exactly $\{0, 1, 2\}$.

Problem 2 (Bernoulli Competition 2023)

Let e be Euler's number. Show that for any odd prime p, the integer

$$1! + 2! + 3! + \cdots + (p-1)! - \left\lfloor \frac{(p-1)!}{e} \right\rfloor$$

is divisible by p.

Solution:

First note that:

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = \sum_{i=0}^{+\infty} \frac{(-1)^i}{i!}.$$

Thus, we have

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor = \left\lfloor \sum_{i=0}^{+\infty} \frac{(-1)^i (p-1)!}{i!} \right\rfloor$$

Notice that:

$$\sum_{i=0}^{p-2} \frac{(-1)^i (p-1)!}{i!} \in \mathbb{Z}$$

We argue that the tail $\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} \in]0;1[$, indeed since p is odd:

$$\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} = \sum_{j=0}^{+\infty} \left(\frac{(p-1)!}{(p-1+2j)!} - \frac{(p-1)!}{(p+2j)!} \right).$$

is certainly bigger than 0 since each term $\left(\frac{(p-1)!}{(p-1+2j)!} - \frac{(p-1)!}{(p+2j)!} \right) = \frac{(p-1)!}{(p-1+2j)!} \left(1 - \frac{1}{p+2j}\right) > 0$. Similarly since p is odd:

$$\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} = 1 - \sum_{j=0}^{\infty} \left(\frac{(p-1)!}{(p+2j)!} - \frac{(p-1)!}{(p+2j+1)!} \right).$$

is certainly smaller than 1 since each term $\left(\frac{(p-1)!}{(p+2j)!} - \frac{(p-1)!}{(p+2j+1)!} \right) = \frac{(p-1)!}{(p+2j)!}(1 - \frac{1}{p+2j+1}) > 0$.

Therefore,

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor = \sum_{i=0}^{p-2} \frac{(-1)^i (p-1)!}{i!}.$$

Now note that for each $0 \le j \le p-1$ we have $j \equiv -(p-j) \pmod{p}$ and thus for fixed $0 \le i < p-1$:

$$\frac{(-1)^i (p-1)!}{i!} = (-1)^i (i+1)(i+2)\cdots(p-1)$$
$$\equiv (-1)^i (p-(i+1))(p-(i+2))\cdots 2 \cdot 1 \cdot (-1)^{p-(i+1)} \equiv (p-(i+1))! \pmod{p},$$

where we used that there is $p-(i+1)$ factor in $\frac{(p-1)!}{i!}$ and the fact that p is odd again. Hence, we have

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor \equiv \sum_{i=0}^{p-2} (p-(i+1))! \equiv \sum_{i=1}^{p-1} i! \pmod{p},$$

since $i \mapsto p-(i+1)$ is a bijection from $[\![0, p-2]\!]$ to $[\![1, p-1]\!]$. This shows the problem's statement.

Problem in example page 140 (PUTNAM and BEYOND)

Find all real solutions to the equation

$$4^x + 6^{x^2} = 5^x + 5^{x^2}.$$

Solution:

Note that $x = 0$ and $x = 1$ satisfy the equation from the statement. Are there other solutions? The answer is no, but to prove it we use the amazing idea of treating the numbers 4, 5, 6 as variables and the presumably new solution x as a constant.

Thus let us consider the function $f(t) = t^{x^2} + (10 - t)^x$. The fact that x satisfies the equation from the statement translates to $f(5) = f(6)$. By Rolle's theorem there exists $c \in (5, 6)$, such that $f'(c) = 0$. This means that

$$x^2 c^{x^2-1} - x(10 - c)^{x-1} = 0,$$

or

$$xc^{x^2-1} = (10 - c)^{x-1}.$$

Because exponentials are positive, this implies that x is positive.
If $x > 1$, then $x^2 - 1 > x - 1$ and as $c > 5$

$$(10 - c)^{x-1} = xc^{x^2-1} > c^{x^2-1} > c^{x-1} > (10 - c)^{x-1},$$

which is a contradiction.

If $0 < x < 1$, then $x^2 - 1 < x - 1$ and:

$$(10 - c)^{x-1} = xc^{x^2-1} < xc^{x-1}.$$

Let us prove that

$$xc^{x-1} < (10 - c)^{x-1}.$$

With the substitution $y = x - 1$, the inequality can be rewritten as

$$y + 1 < \left(\frac{10 - c}{c}\right)^y.$$

which must be proven for $y \in]-1; 0[$.

Lets make a simple analysis of the two functions defined over $\mathbb{R}$.
The exponential has base less than 1, so it is strictly decreasing, while the affine function on the left is strictly increasing. The two meet at $y = 0$ so we must have that strictly before $y = 0$ the exponential is strictly bigger than the affine. The inequality (on $]-1; 0[$) follows. Using it we conclude again that: $(10 - c)^{x-1} = xc^{x^2-1} < (10 - c)^{x-1}$ which is a contradiction. This shows that a third solution to the equation from the statement does not exist. So the only solutions to the given equation are $x = 0$ and $x = 1$.

Problem 3 (Bernoulli Competition 2023)

Let $n \geq 1$ and A be a $n \times n$ symmetric matrix over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ with $1_{\mathbb{F}_2}$'s on the main diagonal. Show that the vector composed uniquely of $1_{\mathbb{F}_2}$'s is in the image of A.

Solution:

We write $1 := 1_{\mathbb{F}_2}$ and $0 := 0_{\mathbb{F}_2}$, define the $\mathbb{F}_2-$vector space $V = \mathbb{F}_2^n$ and define the standard binary product on V, i.e.,

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i.$$

It is easy to see that $\langle \cdot, \cdot\rangle$ is $\mathbb{F}_2-$linear in the first coordinate, symmetric (so $\mathbb{F}_2-$linear in the second coordinate and thus $\mathbb{F}_2-$bilinear) and non-degenerate that is:

$$\forall v \in V\,((\forall w \in V \langle v, w\rangle = 0) \to v = 0_V)$$

(*just plug the canonical basis for w that is for each $i \in n$ take $w = e_i$ and use the fact that $v_i \cdot 1_{\mathbb{F}_2} = v_i$ to conclude $v_i = 0$, and thus $v = \underline{0} = 0_V$*).

With this being introduced, lets take an $n \times n$-matrix A with $\mathrm{diag}(A) = \underline{1}$. Since $\langle \cdot, \cdot\rangle$ is symmetric and non-degenerate, we have for any $\mathbb{F}_2-$subspace $W \subset V$,

$$\left(W^{\perp}\right)^{\perp} = W.$$

where $Z^{\perp} = \{v \in V \mid \langle v, z\rangle = 0\ \forall z \in Z\}$ for any $\mathbb{F}_2-$subspace $Z \subset V$. For more information on this equality see the Appendix [A]. Now write $A = (a_{ij})_{1\leq i,j\leq n}$ with $a_{ii} = 1$ and $a_{ij} = a_{ji}$. Then we have for any $v \in V$,

$$\langle v, Av\rangle = \sum_{1\leq i,j\leq n} v_i v_j a_{ij} = \sum_{i=1}^{n} v_i^2 + 2\sum_{i<j} v_i v_j a_{ij} = \sum_{i=1}^{n} v_i = \langle v, \underline{1}\rangle,$$

because we are working over $\mathbb{F}_2$. In particular for any $z \in \mathrm{Im}(A)^{\perp} \subset V$,

$$\langle z, \underline{1}\rangle = \langle z, Az\rangle = 0,$$

since $Az \in \mathrm{Im}(A)$ and $z \in Im(A)^{\perp}$. As $z \in Im(A)^{\perp}$ was arbitrary, we must have

$$\underline{1} \in \left(\mathrm{Im}(A)^{\perp}\right)^{\perp} = \mathrm{Im}(A).$$

Problem (unknown)

Find all differentiable functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ having at least one fixed point $\alpha \in \mathbb{R}_{>0}$ satisfying:

$$f' = \frac{f}{f \circ f}.$$

Solution:

The identity function obviously works. We claim this is the only solution. Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a differentiable function with a fixed point $\alpha > 0$ and satisfying:

$$f' = \frac{f}{f \circ f}.$$

Note that f is continuous (since f is differentiable), therefore we can integrate it and define for any $x > 0$:

$$F(x) := \int_{\alpha}^{x} f(t)\,\mathrm{d}t.$$

The first fundamental theorem of calculus gives us easily that $F : \mathbb{R}_{>0} \to \mathbb{R}$ is continuous. Moreover, since f is continuous, F is everywhere differentiable with $F' = f$ (this holds for both $x \geq \alpha$ and $0 < x < \alpha$). Because f is always strictly positive, F is strictly increasing and thus injective.

Fix $x \in \mathbb{R}_{>0}$. From the given equation (valid for all $t > 0$),

$$f(t) = f(f(t))f'(t) = F'(f(t))f'(t) = (F \circ f)'(t),$$

we obtain continuity of $(F \circ f)'$ and so its integrability, thus:

$$F(x) = \int_{\alpha}^{x} f(t)\,\mathrm{d}t = \int_{\alpha}^{x} (F \circ f)'(t)\,\mathrm{d}t = (F \circ f)(x) - (F \circ f)(\alpha),$$

where we used the second fundamental theorem of calculus for the function $(F \circ f)'$ (this holds for both $x \geq \alpha$ and $0 < x < \alpha$).

Now, using the fact that α is a fixed point of f, we get $F(f(\alpha)) = F(\alpha) = 0$, so $F(x) = F(f(x))$. By the injectivity of F, we conclude that $f(x) = x$. Since $x > 0$ was arbitrary, the proof is complete.

Bonus: What happens if f has no fixed point?

Solution:

Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a differentiable function satisfying:

$$f' = \frac{f}{f \circ f}$$

such that f has **no** fixed point. Note that f takes positive values, so we must have that $f' = \frac{f}{f \circ f}$ takes strictly positive values. Therefore, f must be strictly increasing and hence injective. From this, we derive the equivalence $\forall \beta > 0$:

$$f'(\beta) = 1 \Leftrightarrow f(\beta) = f(f(\beta)) \Leftrightarrow f(\beta) = \beta.$$

Thus, the existence of a fixed point $\beta > 0$ is equivalent to the fact that $f'(\beta) = 1$. Since f has no fixed point, f' can never take the value 1. As we have seen in the first part, f is continuous, so must be $f \circ f$, and thus f' is continuous (being the quotient of the continuous function f with $f \circ f$). Knowing this, we infer that we cannot have one value of f' strictly bigger than 1 and one value strictly less than 1; otherwise, by the intermediate value theorem (or simply because the image of a connected set is connected), we would have 1 as a value of f'. Hence, we must be in two cases:

$$\text{either } \forall x > 0,\ f'(x) > 1 \text{ or } \forall x > 0,\ f'(x) < 1.$$

• If $\forall x > 0,\ f'(x) > 1$, then in particular, $f'(1) > 1$, so $f(1) > f(f(1))$. Since f is strictly increasing, we must have $1 > f(1)$ (if $1 \leq f(1)$, then $f(1) \leq f(f(1))$, so $f(1) < f(1)$, a contradiction). Since f' is continuous, it is integrable over any compact interval in $\mathbb{R}_{>0}$. Because $\forall t > 0,\ f'(t) > 1$, we have for any $0 < x < 1$:

$$f(1) - f(x) = \int_x^1 f'(t)\,\mathrm{d}t \geq \int_x^1 1\,\mathrm{d}t = 1 - x$$

where we used the second fundamental theorem of calculus for the functions f' and $\underline{1}$ and the fact $\int_x^1 \_\mathrm{d}t$ is increasing from the space of real-valued integrable functions over $[x, 1]$.

Thus, $\forall x > 0$ with $x < 1$,

$$f(x) \leq (f(1) - 1) + x.$$

But then, for any $0 < y < 1 - f(1) < 1$, we have $f(y) < 0$, contradicting the positivity of f.

• This means we must be in the latter case $\forall x > 0,\ f'(x) < 1$. Here the problem becomes significantly more challenging. Since $f'(x)$ is determined by the value of f at x and its composition $f \circ f$ at x, and because $f(f(x)) > f(x)$ (as $f'(x) < 1$), the derivative $f'(x)$ depends on values of f at points beyond $f(x)$. This leads to a non-causal delay differential equation. We will classify all differentiable functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ satisfying $\forall x \in \mathbb{R}_{>0}\ f'(x) < 1$ and:

$$f' = \frac{f}{f \circ f}.$$

Not done yet; I have some incomplete arguments. See the related MathStack Exchange thread
If you have a solution, send them to me:
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temptative Define as in the first part for $x > 0$ (since f is continuous): $F(x) = \int_1^x f(t)\mathrm{d}t$. Since f is always strictly positive, F is strictly increasing and thus injective, therefore invertible in its image $F[\mathbb{R}_{>0}]$. Denote by $g : F[\mathbb{R}_{>0}] \to \mathbb{R}_{>0}$ its inverse. Then g is derivable **WHY???**. Moreover g'=

Theorem 1 (Transformation to Delay Equation)**.** *Let f satisfy the functional equation with $\exists x : f'(x) < 1$. Define $F(x) = \int_1^x f(t)dt$ and $g = F^{-1}$. Then g satisfies:*

$$g'(x) = \frac{1}{g(x+1)} \quad \textit{with} \quad g(0) = 0.$$

Proof. **Step 1: Injectivity.** Since $f > 0$, F is strictly increasing and invertible. Let $g = F^{-1}$.

Step 2: Differentiation. By inverse function theorem:

$$g'(x) = \frac{1}{f(g(x))}.$$

Step 3: Functional Relation. From the original equation:

$$f(f(x))f'(x) = f(x) \implies (F \circ f)'(x) = f(x).$$

Integrate from α to x:

$$F(f(x)) - F(f(\alpha)) = F(x) - F(\alpha).$$

Absorb constants by translation to get $F(f(x)) = F(x) + 1$.

Step 4: Delay Equation. Differentiate $F(f(x)) = F(x) + 1$:

$$f(f(x))f'(x) = f(x) \implies f'(x) = \frac{f(x)}{f(f(x))} = \frac{1}{g(F(x)+2)}.$$

Substitute $f(x) = g(F(x) + 1)$:

$$g'(F(x)+1) = \frac{1}{g(F(x)+2)} \implies g'(x) = \frac{1}{g(x+1)}.$$

□

1 Constructing Solutions

Theorem 2 (Existence and Uniqueness)**.** *There exists a unique global solution $g : \mathbb{R} \to \mathbb{R}_{>0}$ to the delay equation $g'(x) = \frac{1}{g(x+1)}$ with $g(0) = 0$.*

Proof. **Base Case:** Define g on $[0, 1]$ as any $\mathcal{C}^1$-function with:

$$g(0) = 0, \quad g(1) = \beta > 0, \quad g'(0^+) = \infty.$$

Forward Construction ($x \geq 1$): For $x \in [n, n+1]$:

$$g'(x) = \frac{1}{g(x-n+1)}.$$

Integrate recursively using known values from $[n-1, n]$.

Backward Construction ($x < 0$): For $x \in [-n, -n+1]$:

$$g(x) = \int_0^x \frac{1}{g(t+1)} dt.$$

Solve sequentially from $n = 1$ using prior intervals.

Monotonicity: By induction:

- If g is increasing on $[k, k+1]$, then $g' > 0$ on $[k+1, k+2]$
- Backward solutions inherit monotonicity from forward terms

Asymptotics: For large x, approximate:

$$g(x) \approx \sqrt{2x} \quad \text{since} \quad \frac{d}{dx}\sqrt{2x} = \frac{1}{\sqrt{2x}} \approx \frac{1}{\sqrt{2(x+1)}}.$$

Uniqueness: Follows from Lipschitz continuity in g on bounded intervals and recursive determination. □

2 Recovering Original Functions

Theorem 3 (Solution Family)**.** *All solutions with $f'(x) < 1$ are scaling transformations:*

$$f_a(x) = a \cdot g\left(\frac{1}{a^2}\int_{\alpha}^{x/a} g^{-1}(t)dt + 1\right)$$

for $a > 0$, where g solves Theorem 1.

Proof. **Inversion:** Given g, define $F(x) = \int_\alpha^x f(t)dt$. Then:

$$f(x) = g(F(x) + 1).$$

Scaling Invariance: Let $f_a(x) = a \cdot f(x/a)$. Then:

$$f'_a(x) = f'(x/a) = \frac{f(x/a)}{f(f(x/a))} = \frac{f_a(x)}{f_a(f_a(x))}.$$

Thus f_a satisfies the original equation.

Parameterization: The scaling parameter a generates distinct solutions through:

$$g_a(x) = a \cdot g\left(\frac{x}{a^2}\right).$$

□

Remark. The fundamental solution g exhibits delayed dependence, making the system non-Markovian. For numerical constructions, see this interactive graph. We thank everyone that participate in the related MathStack Exchange thread.

A

Let E be a finite-dimensional vector space over a field K, and $B : E \times E \to K$ a symmetric bilinear form. For any subspace $Q \subseteq E$, the orthogonal complement is defined as $Q^{\perp} := \{v \in E \mid \forall q \in Q,\, B(q, v) = 0\}$. It is clearly a K-subspace of E. The form B is non-degenerate if $E^{\perp} = \{0\}$. For a non-degenerate symmetric bilinear form B, the map

$$\varphi : E \to E^{\vee}, \quad v \mapsto B(\cdot, v)$$

is a vector space isomorphism. Indeed, B is bilinear, so φ is linear. The injectivity follows from

$$\varphi(v) = 0_{E^{\vee}} \implies \forall w \in E,\, B(w, v) = 0 \implies v \in E^{\perp} = \{0\} \quad \text{(by non-degeneracy)}.$$

As $\dim(E) = \dim(E^{\vee})$ in finite dimensions (classical), injectivity implies surjectivity. Thus, φ is an isomorphism.

Theorem 4 (Double Orthogonal Complement)**.** *In this setting, if B is non-degenerate and $Q \subseteq E$ is a K-subspace, then $\left(Q^{\perp}\right)^{\perp} = Q$.*

Proof. **Step 1.** $Q \subseteq \left(Q^{\perp}\right)^{\perp}$: Let $q \in Q$, then,

$$\forall v \in Q^{\perp},\, 0 = B(q, v) = B(v, q) \implies q \in \left(Q^{\perp}\right)^{\perp}.$$

Step 2. Using the isomorphism φ, we have that $\varphi|_{Q^{\perp}}$ is an isomorphism onto its image:

$$\operatorname{Im}\left(\varphi|_{Q^{\perp}}\right) = \left\{B(-, v) \mid v \in Q^{\perp}\right\} = \{f \in E^{\vee} \mid \forall q \in Q,\, f(q) = 0\} =: Q^{\circ}.$$

The middle equality's $\supset$ inclusion follows from the surjectivity of φ and the definition of $Q^{\perp}$. Thus, $\dim\left(Q^{\perp}\right) = \dim(Q^{\circ})$.

Step 3. Given an ordered basis $\langle w_i \rangle_{i \in \dim(Q)}$ of Q, complete it into an ordered basis of E:

$$\langle w_i \rangle_{i \in \dim(Q)} \frown \langle v_j \rangle_{j \in \dim(E) - \dim(Q)}.$$

There is a classical associated ordered basis of $E^{\vee}$:

$$\langle w_i^* \rangle_{i \in \dim(Q)} \frown \left\langle v_j^* \right\rangle_{j \in \dim(E) - \dim(Q)},$$

where each functional sends a vector $v = \sum_{j \in \dim(Q)} \lambda_j w_j + \sum_{i \in \dim(E) - \dim(Q)} \gamma_i v_i$ of E to λ_j or γ_i, respectively. This basis satisfies the following property: if $f \in E^{\vee}$, then we can write

$$f = \sum_{i \in \dim(Q)} f(w_i)\, w_i^* + \sum_{j \in \dim(E) - \dim(Q)} f(v_j)\, v_j^*.$$

In particular, if $f \in Q^{\circ}$, then $f = \sum_{j \in \dim(E) - \dim(Q)} f(v_j)\, v_j^*$, because $\{w_i \mid i \in \dim(Q)\} \subset Q$. Thus, $\left\langle v_j^* \right\rangle_{j \in \dim(E) - \dim(Q)}$ generates Q°. Since these vectors are K-linearly independent, we obtain

$$\dim(E) - \dim(Q) = \dim(Q^{\circ}).$$

Step 4. In total, we obtain that for any K-subspace Q of E,

$$\dim\left(Q^{\perp}\right) = \dim(Q^{\circ}) = \dim(E) - \dim(Q).$$

Since $Q^\perp$ is a K-subspace, we must have:

$$\dim\left(\left(Q^\perp\right)^\perp\right) = \dim(E) - \dim\left(Q^\perp\right) = \dim(E) - (\dim(E) - \dim(Q)) = \dim(Q).$$

We conclude $Q = \left(Q^\perp\right)^\perp$ since $Q \subset \left(Q^\perp\right)^\perp$ and they have the same (crucially finite) dimension. $\square$