Problem Set Week 4 Solutions

Math Olympiad Club Zurich

Spring 2025

Problem: unknown

Find all real solutions to the equation

$$9^x + 4^x + 2^x = 8^x + 6^x + 1.$$

Solution:

It is easy to see that $x = 0$, $x = 1$, and $x = 2$ are solutions. So the equation has at least 3 distinct real solutions. Let us introduce the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 9^x + 4^x + 2^x - 8^x + 6^x + 1.$$

As stated above, f has at least 3 distinct zeros. We claim there are no other roots. By Rolle's theorem, if a function $g : \mathbb{R} \to \mathbb{R}$ has at least $n \geq 2$ zeros $x_1 < \cdots < x_n$, then the function $g'(x)$ has at least $n-1$ zeros $y_1 < \cdots < y_{n-1}$, where for each $i = 1, \ldots, n-1$, we have $x_i < y_i < x_{i+1}$. In particular, since for each $a \in \mathbb{R}_{>0}$, $a^{-x}g(x)$ has at least n zeros, we have that $\left(a^{-x}g(x)\right)'$ has at least $n-1$ zeros, and so does the function

$$h_a g(x) = a^x \left(a^{-x}g(x)\right)' = g'(x) - \ln(a)g(x).$$

Suppose f has another zero not in $\{0, 1, 2\}$. Then f has at least 4 zeros, and thus

$$h_1 f(x) = f'(x) = \ln(9)9^x + \ln(4)4^x + \ln(2)2^x - \ln(8)8^x - \ln(6)6^x$$

has at least 3 zeros, which then implies that

$$h_6 h_1 f(x) = f''(x) - \ln(6)f'(x)$$

$$= \ln\left(\frac{9}{6}\right)\ln(9)9^x + \ln\left(\frac{4}{6}\right)\ln(4)4^x + \ln\left(\frac{2}{6}\right)\ln(2)2^x - \ln\left(\frac{8}{6}\right)\ln(8)8^x$$

has at least 2 zeros, which again implies that

$$h_8 h_6 h_1 f(x) = \ln\left(\frac{9}{8}\right)\ln\left(\frac{9}{6}\right)\ln(9)9^x - \ln\left(\frac{4}{8}\right)\ln\left(\frac{4}{6}\right)\ln(4)4^x - \ln\left(\frac{2}{8}\right)\ln\left(\frac{2}{6}\right)\ln(2)2^x$$

has at least 1 zero.

The function $h_8 h_6 h_1 f(x)$ is of the form $k_2 2^x + k_4 4^x + k_9 9^x$, for $k_2, k_4, k_9 > 0$ and hence is always positive. Therefore, $h_8 h_6 h_1 f(x)$ cannot have any real zero. This is a contradiction to the assumptions hence the solutions of the original equation are exactly $\{0, 1, 2\}$.

Problem: 2 Bernoulli Competition 2023

Let e be Euler's number. Show that for any odd prime p, the integer

$$1! + 2! + 3! + \cdots + (p-1)! - \left\lfloor \frac{(p-1)!}{e} \right\rfloor$$

is divisible by p.

Solution:

First note that:

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = \sum_{i=0}^{+\infty} \frac{(-1)^i}{i!}.$$

Thus, we have

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor = \left\lfloor \sum_{i=0}^{+\infty} \frac{(-1)^i (p-1)!}{i!} \right\rfloor$$

Notice that:

$$\sum_{i=0}^{p-2} \frac{(-1)^i (p-1)!}{i!} \in \mathbb{Z}$$

We argue that the tail $\sum_{i=p-1}^{+\infty} \frac{(-1)^i(p-1)!}{i!} \in]0;1[$, indeed since p is odd:

$$\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} = \sum_{j=0}^{+\infty} \left(\frac{(p-1)!}{(p-1+2j)!} - \frac{(p-1)!}{(p+2j)!} \right).$$

is certainly bigger than 0 since each term $\left(\frac{(p-1)!}{(p-1+2j)!} - \frac{(p-1)!}{(p+2j)!}\right) = \frac{(p-1)!}{(p-1+2j)!}\left(1 - \frac{1}{p+2j}\right) > 0$. Similarly since p is odd:

$$\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} = 1 - \sum_{j=0}^{\infty} \left(\frac{(p-1)!}{(p+2j)!} - \frac{(p-1)!}{(p+2j+1)!} \right).$$

is certainly smaller than 1 since each term $\left(\frac{(p-1)!}{(p+2j)!} - \frac{(p-1)!}{(p+2j+1)!}\right) = \frac{(p-1)!}{(p+2j)!}(1 - \frac{1}{p+2j+1}) > 0$.

Therefore,

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor = \sum_{i=0}^{p-2} \frac{(-1)^i (p-1)!}{i!}.$$

Now note that for each $0 \le j \le p-1$ we have $j \equiv -(p-j) \pmod{p}$ and thus for fixed $0 \le i < p-1$:

$$\frac{(-1)^i (p-1)!}{i!} = (-1)^i (i+1)(i+2)\cdots(p-1)$$

$$\equiv (-1)^i (p-(i+1))(p-(i+2))\cdots 2 \cdot 1 \cdot (-1)^{p-(i+1)} \equiv (p-(i+1))! \pmod{p},$$

where we used that there is $p-(i+1)$ factor in $\frac{(p-1)!}{i!}$ and the fact that p is odd again. Hence, we have

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor \equiv \sum_{i=0}^{p-2} (p-(i+1))! \equiv \sum_{i=1}^{p-1} i! \pmod{p},$$

since $i \mapsto p-(i+1)$ is a bijection from $[\![0, p-2]\!]$ to $[\![1, p-1]\!]$. This shows the problem's statement.

Problem: Example p.140 PUTNAM and BEYOND

Find all real solutions to the equation

$$4^x + 6^{x^2} = 5^x + 5^{x^2}.$$

Solution:

Note that $x = 0$ and $x = 1$ satisfy the equation from the statement. Are there other solutions? The answer is no, but to prove it we use the amazing idea of treating the numbers 4, 5, 6 as variables and the presumably new solution x as a constant.

Thus let us consider the function $f(t) = t^{x^2} + (10 - t)^x$. The fact that x satisfies the equation from the statement translates to $f(5) = f(6)$. By Rolle's theorem there exists $c \in (5, 6)$, such that $f'(c) = 0$. This means that

$$x^2 c^{x^2-1} - x(10 - c)^{x-1} = 0,$$

or

$$xc^{x^2-1} = (10 - c)^{x-1}.$$

Because exponentials are positive, this implies that x is positive.
If $x > 1$, then $x^2 - 1 > x - 1$ and as $c > 5$

$$(10 - c)^{x-1} = xc^{x^2-1} > c^{x^2-1} > c^{x-1} > (10 - c)^{x-1},$$

which is a contradiction.

If $0 < x < 1$, then $x^2 - 1 < x - 1$ and:

$$(10 - c)^{x-1} = xc^{x^2-1} < xc^{x-1}.$$

Let us prove that

$$xc^{x-1} < (10 - c)^{x-1}.$$

With the substitution $y = x - 1$, the inequality can be rewritten as

$$y + 1 < \left(\frac{10 - c}{c}\right)^y.$$

which must be proven for $y \in]-1; 0[$.

Lets make a simple analysis of the two functions defined over $\mathbb{R}$.
The exponential has base less than 1, so it is strictly decreasing, while the affine function on the left is strictly increasing. The two meet at $y = 0$ so we must have that strictly before $y = 0$ the exponential is strictly bigger than the affine. The inequality (on $]-1; 0[$) follows. Using it we conclude again that: $(10 - c)^{x-1} = xc^{x^2-1} < (10 - c)^{x-1}$ which is a contradiction. This shows that a third solution to the equation from the statement does not exist. So the only solutions to the given equation are $x = 0$ and $x = 1$.

Problem: 3 Bernoulli Competition 2023

Let $n \geq 1$ and A be a $n \times n$ symmetric matrix over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ with $1_{\mathbb{F}_2}$'s on the main diagonal. Show that the vector composed uniquely of $1_{\mathbb{F}_2}$'s is in the image of A.

Solution:

Define the standard binary product on the finite dimensional $\mathbb{F}_2$-vector space $\mathbb{F}_2^n$, i.e.,

$$\langle v, w \rangle_{\mathbb{F}_2^n} = \sum_{i=0}^{n-1} v_i w_i.$$

It is easy to see that $\langle \cdot, \cdot \rangle_{\mathbb{F}_2^n}$ is $\mathbb{F}_2$-bilinear, symmetric and non-degenerate that is:

$$\forall v \in \mathbb{F}_2^n \left(\left(\forall w \in \mathbb{F}_2^n \langle v, w \rangle_{\mathbb{F}_2^n} = 0_{\mathbb{F}_2} \right) \to v = 0_{\mathbb{F}_2^n} \right)$$

Since $\langle \cdot, \cdot \rangle_{\mathbb{F}_2^n}$ is symmetric and non-degenerate, we have for any $\mathbb{F}_2-$subspace $W \subset \mathbb{F}_2^n$,

$$\left(W^{\perp} \right)^{\perp} = W.$$

where $Z^{\perp} = \{ v \in \mathbb{F}_2^n \mid \langle v, z \rangle_{\mathbb{F}_2^n} = 0_{\mathbb{F}_2} \ \forall z \in Z \}$ for any $\mathbb{F}_2-$subspace $Z \subset \mathbb{F}_2^n$. For a proof of this classical fact, see the Appendix [A].

With this being introduced, lets take an $n \times n$-matrix A with $\operatorname{diag}(A) = \underline{1}$. Now write $A = (a_{ij})_{1 \leq i,j \leq n}$ with $a_{ii} = 1_{\mathbb{F}_2}$ and $a_{ij} = a_{ji}$. Then we have for any $v \in \mathbb{F}_2^n$,

$$\langle v, Av \rangle_{\mathbb{F}_2^n} = \sum_{0 \leq i,j \leq n-1} v_i v_j a_{ij} = \sum_{i=1}^{n} v_i^2 + 2 \sum_{0 \leq i < j \leq n-1} v_i v_j a_{ij} = \sum_{i=0}^{n-1} v_i = \left\langle v, 1_{\mathbb{F}_2^n} \right\rangle_{\mathbb{F}_2^n},$$

because we are working over $\mathbb{F}_2$. In particular for any $z \in \operatorname{Im}(A)^{\perp} \subset \mathbb{F}_2^n$,

$$\langle z, \underline{1} \rangle_{\mathbb{F}_2^n} = \langle z, Az \rangle_{\mathbb{F}_2^n} = 0_{\mathbb{F}_2},$$

since $Az \in \operatorname{Im}(A)$ and $z \in \operatorname{Im}(A)^{\perp}$. As $z \in \operatorname{Im}(A)^{\perp}$ was arbitrary, we must have

$$1_{\mathbb{F}_2^n} \in \left(\operatorname{Im}(A)^{\perp} \right)^{\perp} = \operatorname{Im}(A).$$

Problem: unknown

Find all differentiable functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ having at least one fixed point $\alpha \in \mathbb{R}_{>0}$ satisfying:

$$f' = \frac{f}{f \circ f}.$$

Solution:

The identity function obviously works. We claim this is the only solution. Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a differentiable function with a fixed point $\alpha > 0$ and satisfying:

$$f' = \frac{f}{f \circ f}.$$

Note that f is continuous (since f is differentiable), therefore we can integrate it and define for any $x > 0$:

$$F(x) := \int_{\alpha}^{x} f(t)\, \mathrm{d}t.$$

The first fundamental theorem of calculus gives us easily that $F : \mathbb{R}_{>0} \to \mathbb{R}$ is continuous. Moreover, since f is continuous, F is everywhere differentiable with $F' = f$ (this holds for both $x \geq \alpha$ and $0 < x < \alpha$). Because f is always strictly positive, F is strictly increasing and thus injective.

Fix $x \in \mathbb{R}_{>0}$. From the given equation (valid for all $t > 0$),

$$f(t) = f(f(t))f'(t) = F'(f(t))f'(t) = (F \circ f)'(t),$$

we obtain the continuity of $(F \circ f)'$ and so its integrability, thus:

$$F(x) = \int_{\alpha}^{x} f(t)\, \mathrm{d}t = \int_{\alpha}^{x} (F \circ f)'(t)\, \mathrm{d}t = (F \circ f)(x) - (F \circ f)(\alpha),$$

where we used the second fundamental theorem of calculus for the function $(F \circ f)'$ (this holds for both $x \geq \alpha$ and $0 < x < \alpha$).

Now, using the fact that α is a fixed point of f, we get $F(f(\alpha)) = F(\alpha) = 0$, so $F(x) = F(f(x))$. By the injectivity of F, we conclude that $f(x) = x$. Since $x > 0$ was arbitrary, the proof is complete.

Bonus: What happens if f has no fixed point?

Solution:

Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a differentiable function satisfying:

$$f' = \frac{f}{f \circ f}$$

such that f has **no** fixed point. Note that f takes positive values, so we must have that $f' = \frac{f}{f \circ f}$ takes strictly positive values. Therefore, f must be strictly increasing and hence injective. From this, we derive the equivalence $\forall \beta > 0$:

$$f'(\beta) = 1 \Leftrightarrow f(\beta) = f(f(\beta)) \Leftrightarrow f(\beta) = \beta.$$

Thus, the existence of a fixed point $\beta > 0$ is equivalent to the fact that $f'(\beta) = 1$. Since f has no fixed point, f' can never take the value 1. As we have seen in the first part, f is continuous, so must be $f \circ f$, and thus f' is continuous (being the quotient of the continuous function f with $f \circ f$). Knowing this, we infer that we cannot have one value of f' strictly bigger than 1 and one value strictly less than 1; otherwise, by the intermediate value theorem (or simply because the image of a connected set is connected), we would have 1 as a value of f'. Hence, we must be in two cases:

$$\text{either } \forall x > 0,\ f'(x) > 1 \text{ or } \forall x > 0,\ f'(x) < 1.$$

• If $\forall x > 0,\ f'(x) > 1$, then in particular, $f'(1) > 1$, so $f(1) > f(f(1))$. Since f is strictly increasing, we must have $1 > f(1)$ (if $1 \leq f(1)$, then $f(1) \leq f(f(1))$, so $f(1) < f(1)$, a contradiction). Since f' is continuous, it is integrable over any compact interval in $\mathbb{R}_{>0}$. Because $\forall t > 0,\ f'(t) > 1$, we have for any $0 < x < 1$:

$$f(1) - f(x) = \int_x^1 f'(t)\,\mathrm{d}t \geq \int_x^1 1\,\mathrm{d}t = 1 - x$$

where we used the second fundamental theorem of calculus for the functions f' and $\underline{1}$ and the fact $\int_x^1 \_\mathrm{d}t$ is increasing from the space of real-valued integrable functions over $[x, 1]$.

Thus, $\forall x > 0$ with $x < 1$,

$$f(x) \leq (f(1) - 1) + x.$$

But then, for any $0 < y < 1 - f(1) < 1$, we have $f(y) < 0$, contradicting the positivity of f.

• This means we must be in the latter case $\forall x > 0,\ f'(x) < 1$. Here the problem becomes significantly more challenging. Since $f'(x)$ is determined by the value of f at x and its composition $f \circ f$ at x, and because $f(f(x)) > f(x)$ (as $f'(x) < 1$), the derivative $f'(x)$ depends on values of f at points beyond $f(x)$. This leads to a non-causal delay differential equation. We will classify all differentiable functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ satisfying $\forall x \in \mathbb{R}_{>0}\ f'(x) < 1$ and:

$$f' = \frac{f}{f \circ f}.$$

Not done yet; I have some incomplete arguments. See the related MathStack Exchange thread
If you have a solution, send them to me:
antoine@du−fresne.ch

Remark. The fundamental solution exhibits delayed dependence, making the system non-Markovian. For numerical constructions, see this interactive graph. We thank everyone that participate in the related MathStack Exchange thread.

A

Let E be a vector space over a field K, and let $B\colon E \times E \to K$ be a symmetric K-bilinear form. For any subspace $Q \subseteq E$, the orthogonal complement is defined by

$$Q^{\perp} := \{v \in E \mid \forall q \in Q,\, B(q, v) = 0_K\}.$$

It is clearly a K-subspace of E. The map

$$\varphi_Q\colon E \to Q^*, \quad v \mapsto B(\cdot, v)\big|_Q$$

is K-linear (since B is K-bilinear) and has kernel $Q^{\perp}$, because

$$v \in \ker(\varphi_Q) \iff \varphi_Q(v) = 0_{Q^*} \iff \forall w \in Q,\, B(w, v) = 0_K \iff v \in Q^{\perp}.$$

If $\dim(Q)$ is finite, then $\dim(Q) = \dim(Q^*)$[1]. Hence, by the rank–nullity theorem, we obtain a standard general inequality

$$\dim(E) = \dim(\ker(\varphi_Q)) + \dim(\operatorname{Im}(\varphi_Q))$$

$$= \dim\left(Q^{\perp}\right) + \dim(\operatorname{Im}(\varphi_Q)) \leq \dim\left(Q^{\perp}\right) + \dim(Q^*) = \dim\left(Q^{\perp}\right) + \dim(Q).$$

The form B is non-degenerate if $E^{\perp} = \{0_E\}$. Hence φ_E is injective if and only if $E^{\perp} = \{0_E\}$, that is, if and only if B is non-degenerate. If B is non-degenerate and $\dim(E)$ is finite, then φ_E is an isomorphism. Indeed, since $\dim(E) = \dim(E^*)$, injectivity of φ_E implies its surjectivity.

Examples: For any $n \in \mathbb{N}_{\geq 1}$, we define on the finite dimensional K-vector space K^n the standard binary product $\langle \cdot, \cdot \rangle_{K^n} : K^n \times K^n \to K$, i.e.,

$$\langle v, w \rangle_{K^n} = \sum_{i=0}^{n-1} v_i w_i.$$

It is easy to see that $\langle \cdot, \cdot \rangle_{K^n}$ is K-linear in the first coordinate, symmetric (so K-linear in the second coordinate and thus K-bilinear) and non-degenerate that is:

$$\forall v \in K^n \left((\forall w \in K^n \langle v, w \rangle_{K^n} = 0_K) \to v = 0_{K^n}\right)$$

(Just plug the canonical basis for w that is for each $i \in n$ take $w = e_i$ and use the fact that $v_i \cdot 1_K = v_i$ to conclude $v_i = 0_K$, and thus $v = 0_{K^n}$).

Theorem 1 (Double Orthogonal Complement)**.** *In this setting, if* $\dim(E)$ *is finite,* B *is non-degenerate, then*

1. *For any* K*-subspace* $Q \subseteq E$*;* $\dim(E) = \dim\left(Q^{\perp}\right) + \dim(Q)$.
2. *For any* K*-subspace* $Q \subseteq E$*;* $\left(Q^{\perp}\right)^{\perp} = Q$.

Proof. **1.** Using the fact that φ_E is an isomorphism, we have that $(\varphi_E)|_{Q^{\perp}}$ is an isomorphism onto its image:

$$\operatorname{Im}\left((\varphi_E)|_{Q^{\perp}}\right) = \left\{B(-, v) \mid v \in Q^{\perp}\right\} = \{f \in E^* \mid \forall q \in Q,\, f(q) = 0\} =: Q^{\circ}.$$

[1] Given an ordered basis $\langle w_i \rangle_{i \in \dim(Q)}$ of Q (which must be finite), one has an ordered basis $\langle w_i^* \rangle_{i \in \dim(Q)}$ of Q^*, where each functional $w_i^*\colon Q \to K$ sends a vector $v = \sum_{j \in \dim(Q)} \lambda_j w_j \in Q$ to λ_i.

The middle equality's $\supset$ inclusion follows from the surjectivity of φ_E and the definition of $Q^{\perp}$. Thus, $\dim\left(Q^{\perp}\right) = \dim\left(Q^{\circ}\right)$.

Now, given an ordered basis $\langle w_i \rangle_{i \in \dim(Q)}$ of Q, complete it into an ordered basis of E:

$$\langle w_i \rangle_{i \in \dim(Q)} \frown \langle v_j \rangle_{j \in \dim(E) - \dim(Q)} .$$

Then:

$$\langle w_i^* \rangle_{i \in \dim(Q)} \frown \left\langle v_j^* \right\rangle_{j \in \dim(E) - \dim(Q)} ,$$

where each functional sends a vector $v = \sum_{j \in \dim(Q)} \lambda_j w_j + \sum_{i \in \dim(E) - \dim(Q)} \gamma_i v_i$ of E to λ_i or γ_j, respectively, is an ordered basis of E^*. This basis satisfies the following property: if $f \in E^*$, then we can write

$$f = \sum_{i \in \dim(Q)} f\left(w_i\right) w_i^* + \sum_{j \in \dim(E) - \dim(Q)} f\left(v_j\right) v_j^* .$$

In particular, if $f \in Q^{\circ}$, then $f = \sum_{j \in \dim(E) - \dim(Q)} f\left(v_j\right) v_j^*$, because $\{w_i \mid i \in \dim(Q)\} \subset Q$. Thus, $\left\langle v_j^* \right\rangle_{j \in \dim(E) - \dim(Q)}$ generates Q°. Since these vectors are K-linearly independent, we obtain

$$\dim(E) - \dim(Q) = \dim\left(Q^{\circ}\right) .$$

In total, we obtain for any K-subspace Q of E,

$$\dim\left(Q^{\perp}\right) = \dim\left(Q^{\circ}\right) = \dim(E) - \dim(Q) .$$

which gives our desired equality.

2. Let $Q \subseteq \left(Q^{\perp}\right)^{\perp}$: Let $q \in Q$, then,

$$\forall v \in Q^{\perp},\, 0 = B(q, v) = B(v, q) \implies q \in \left(Q^{\perp}\right)^{\perp} .$$

Since $Q^{\perp} \subset E$ is a K-subspace, we must have:

$$\dim\left(\left(Q^{\perp}\right)^{\perp}\right) = \dim(E) - \dim\left(Q^{\perp}\right) = \dim(E) - (\dim(E) - \dim(Q)) = \dim(Q) .$$

We conclude $Q = \left(Q^{\perp}\right)^{\perp}$ since $Q \subset \left(Q^{\perp}\right)^{\perp}$ and they have the same (crucially finite) dimension. $\square$