Problem Set Week 5

ETHZ Math Olympiad Club

24 March 2025

Problem (unknown)

We consider a game where two indistinguishable envelopes are presented to a player:

- One envelope contains an amount $\alpha \in \mathbb{R}_{>0}$.
- The other envelope contains 2α.

The game proceeds as follows:

1. The player randomly selects one envelope (with equal probability).
2. The player observes the content x of the selected envelope (without knowing α).
3. The player must decide whether to:
   - Keep the current envelope, or
   - Switch to the other envelope (this decision is irrevocable).

Although the game is played once, the player's objective is still to maximize their *expected gain*. Assuming access to *randomness*, how can they do better than always keeping the first envelope?

Problem A-3 (IMC 2018)

Determine all rational numbers a for which the matrix

$$A = \begin{bmatrix} a & a & 1 & 0 \\ -a & -a & 0 & 1 \\ -1 & 0 & a & a \\ 0 & -1 & -a & -a \end{bmatrix}$$

is the square of a matrix with all rational entries.

Problem A-4 (IMC 2005)

Find all polynomials

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \quad (a_n \neq 0)$$

satisfying the following two conditions:

1. $(a_0, a_1, \ldots, a_n)$ is a permutation of the numbers $(0, 1, \ldots, n)$, and
2. all roots of $P(X)$ are rational numbers.

Problem A-6 (IMC 2005)

Let $m, n \in \mathbb{Z}$. Given a group G, denote by $G(m)$ the subgroup generated by the m-th powers of elements of G:

$$G(m) := \langle \{g^m \mid g \in G\} \rangle \leq G.$$

If $G(m)$ and $G(n)$ are commutative, prove that $G(\gcd(m, n))$ is also commutative. Here, $\gcd(m, n)$ denotes the greatest common divisor of m and n.