Problem Set Week 6

ETHZ Math Olympiad Club

31 March 2025

Problem B-1 (IMC 2023)

Ivan writes the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$$

on the board. Then he performs the following operation on the matrix several times:

- He chooses a row or a column of the matrix, and
- He multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix

$$B = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$$

after finitely many steps?

Vieta Jumping Problems

0.1 Problem 6 (IMO 1988)

Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.

0.2 Problem (Kevin Buzzard & Edward Crane)

Let a and b be positive integers. Show that if $4ab - 1$ divides $\left(4a^2 - 1\right)^2$, then $a = b$.

Problem A-3 (IMC 2015)

Let $F(0) = 0$, $F(1) = \frac{3}{2}$, and

$$F(n) = \frac{5}{2}F(n-1) - F(n-2) \quad \text{for } n \geq 2.$$

Determine whether or not

$$\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$$

is a rational number.