

# Problem Set Week 8

Math Olympiad Club Zurich

Spring 2025

## Problems: Simon's Favorite Factoring Trick

SFFT is often used in a Diophantine equation where factoring is needed. We let  $R$  be any unitary ring. If we have a multipolynomial in two formal variables  $X$  and  $Y$ ,  $P(X, Y) \in R[X, Y]$  of the form

$$P(X, Y) = aXY + bX + cY + d \in R[X, Y]$$

with  $a \in R^\times$ . According to *Simon's Favorite Factoring Trick*, this multipolynomial is:

$$P(X, Y) = a \left( X + a^{-1}c \right) \left( Y + a^{-1}b \right) + d - ca^{-1}b.$$

### 0.1 AMC 12 (2012)

How many non-congruent right triangles with positive integer leg lengths have areas equal to 3 times their perimeters?

### 0.2 AIME (1998)

An  $m \times n \times p$  rectangular box has half the volume of an  $(m+2) \times (n+2) \times (p+2)$  rectangular box, where  $m, n$ , and  $p$  are integers, and  $m \leq n \leq p$ . What is the largest possible value of  $p$ ?

### 0.3 BMO (2005)

The integer  $N$  is positive. There are exactly 2005 ordered pairs  $(x, y)$  of positive integers satisfying:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$

Prove that  $N$  is a perfect square.

### 0.4 JBMO (2003)

Let  $n$  be a positive integer. A number  $A$  consists of  $2n$  digits, each of which is 4; and a number  $B$  consists of  $n$  digits, each of which is 8. Prove that  $A + 2B + 4$  is a perfect square.

### 0.5 AIME (2000)

The system of equations

$$\begin{aligned} \log_{10}(2000xy) - \log_{10}(x) \log_{10}(y) &= 4, \\ \log_{10}(2yz) - \log_{10}(y) \log_{10}(z) &= 1, \\ \log_{10}(zx) - \log_{10}(z) \log_{10}(x) &= 0, \end{aligned}$$

has two solutions  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ . Find  $y_1 + y_2$ .

## Problem (Wu-Riddles)

Let  $n \in \mathbb{N}_{>0}$  and  $A, B \in \mathbb{R}^{n \times n}$  be real matrices. Now suppose  $I_n - BA$  is invertible, where  $I_n$  is the identity matrix. Prove that  $I_n - AB$  is also invertible.

## Problem (unknown)

An enemy submarine is hidden somewhere along the infinite line  $\mathbb{R}$ . It travels silently, and you know that its path is described by a fixed rational polynomial rule  $\sum_{i=0}^n c_i T^i \in \mathbb{Q}[X]$ ; for each time  $t \in \mathbb{N}$ , its position is given by

$$x(t) = \sum_{i=0}^n c_i t^i \in \mathbb{Q}.$$

However, you do not know  $n \in \mathbb{N}$ , nor the rational coefficients  $(c_i)_{i \in \mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$  that form this rational polynomial rule.

Each unit of time (starting from 0), you are allowed to launch a single torpedo at any chosen rational position. If the submarine is at that position at that time, it is struck and sinks. Assume you possess a potentially countably infinite arsenal of torpedoes and potentially countably infinite time.

Devise a strategy — a sequence of torpedo launches — such that, regardless of the submarine's unknown position, you will **eventually** hit it.

## Problem B1 (Putnam 1960) & A1 (Putnam 1961)

Define

$$A := \{(x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} : x^y = y^x\}^1$$

and find:

$$A, \quad A \cap (\mathbb{Q} \times \mathbb{Q}) \quad \text{and} \quad A \cap (\mathbb{Z} \times \mathbb{Z}).$$

**Bonus:** Find  $A \cap (\mathbb{Q}^{\text{alg}} \times \mathbb{Q}^{\text{alg}})$  and  $A \cap (\mathcal{O}_{\mathbb{Q}^{\text{alg}}}(\mathbb{Z}) \times \mathcal{O}_{\mathbb{Q}^{\text{alg}}}(\mathbb{Z}))$ .

## Problem (Sudakov & Milojević)

Let  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$  be non-constant sequences of rational numbers. Suppose that for all  $i, j \in \mathbb{N}$ ,

$$(a_i - a_j)(b_i - b_j) \in \mathbb{Z}.$$

Prove that there exists a non-zero rational number  $\gamma \in \mathbb{Q}^\times$  such that for all  $i, j \in \mathbb{N}$ ,

$$\gamma(a_i - a_j), \quad \gamma^{-1}(b_i - b_j) \in \mathbb{Z}.$$

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<sup>1</sup>Recall that the exponentiation  $a^b := \exp(b \cdot \log(a))$  where  $a > 0$  and  $b \in \mathbb{R}$  can be extended to  $a = 0$  and  $b \geq 0$  by:

$$0^b = \begin{cases} 0 & (b > 0), \\ 1 & (b = 0). \end{cases}$$

and extended in the obvious way for  $a \in \mathbb{R}^\times$  with  $b \in \mathbb{Z}$ .